

# THE CALABI–YAU EQUATION FOR $T^2$ -BUNDLES OVER $\mathbb{T}^2$ : THE NON-LAGRANGIAN CASE

ERNESTO BUZANO, ANNA FINO, LUIGI VEZZONI

**ABSTRACT.** In the spirit of [9, 2], we study the Calabi-Yau equation on  $T^2$ -bundles over  $\mathbb{T}^2$  endowed with an invariant non-Lagrangian almost-Kähler structure showing that for  $T^2$ -invariant initial data it reduces to a Monge-Ampère equation having a unique solution. In this way we prove that for every total space  $M^4$  of an orientable  $T^2$ -bundle over  $\mathbb{T}^2$  endowed with an invariant almost-Kähler structure the Calabi-Yau problem has a solution for every normalized  $T^2$ -invariant volume form.

## 1. INTRODUCTION

Let  $(M^{2n}, J, \Omega)$  be a  $2n$ -dimensional compact Kähler manifold with associated complex structure  $J$  and symplectic form  $\Omega$ . In view of a celebrated Yau's theorem [12] for every volume form  $\sigma$  on  $M^{2n}$  satisfying

$$(1.1) \quad \int_{M^{2n}} \Omega^n = \int_{M^{2n}} \sigma$$

there exists a unique Kähler form  $\tilde{\Omega}$  in the same de Rham cohomology class of  $\Omega$  and such that

$$(1.2) \quad \tilde{\Omega}^n = \sigma.$$

Equation (1.2) still makes sense in the *almost-Kähler* context when  $J$  is merely an almost-complex structure and  $\Omega$  remains closed. The almost complex structure  $J$  is still orthogonal relative to a Riemannian metric  $g$  for which  $\Omega(X, Y) = g(JX, Y)$ , and

$$(1.3) \quad \tilde{\Omega} = \Omega + d\alpha$$

is again assumed to be a positive-definite  $(1, 1)$ -form relative to  $J$ . In this context the equations (1.1), (1.2) and (1.3) constitute the *Calabi-Yau problem*, which in the last years has been intensively studied in four dimensions (see [1, 8, 9, 2] and the references therein).

In [1] Donaldson introduced the Calabi-Yau problem for almost-Kähler manifolds showing that equation (1.2) has unique solution in dimensions four and that it is related to some other central problems in symplectic geometry. In [8] Tosatti, Weinkove and Yau gave a sufficient condition for the existence of solution to the Calabi-Yau equation in terms of the Chern connection. This condition fails in case of the Kodaira-Thurston surface, which is a 4-dimensional nilmanifold, i.e. a compact quotient of the nilpotent Lie group  $\text{Nil}^3 \times \mathbb{R}$  by a lattice, where  $\text{Nil}^3$  denotes the 3-dimensional real Heisenberg group.

The Kodaira-Thurston surface is a typical example of a compact almost-Kähler 4-dimensional manifold which does not admit any Kähler structure. More precisely, it is the total space of a principal  $T^2$ -bundle over a torus  $\mathbb{T}^2$  (in our notation  $T^2$  denotes the torus on the fibres, while  $\mathbb{T}^2$  is the torus at the basis) and it has an invariant almost-Kähler structure whose symplectic form vanishes along the fibres of the  $T^2$ -fibration, where by invariant structure we mean a structure induced by a left-invariant one on  $\text{Nil}^3 \times \mathbb{R}$ . The almost-Kähler structures on a total space of a fibration whose symplectic form vanishes along the fibres are usually called *Lagrangian*, since the fibers are Lagrangian submanifolds.

In [9] Tosatti and Weinkove studied the Calabi-Yau equation on the Kodaira-Thurston surface endowed with an invariant Lagrangian almost-Kähler structure, showing the existence of a solution for every  $T^2$ -invariant normalized volume form  $\sigma$ . In [2] the previous result obtained by Tosatti and Weinkove was simplified and extended to other  $T^2$ -bundles over a  $\mathbb{T}^2$  endowed with an invariant Lagrangian almost-Kähler structure.

We recall that in view of [10] every orientable  $T^2$ -bundle over a  $\mathbb{T}^2$  is a infra-solvmanifold, i.e. a smooth quotient  $\Gamma \backslash G$  covered by a solvmanifold  $\tilde{\Gamma} \backslash G$ , compact quotient by a co-compact discrete subgroup of one of the following four Lie groups

$$\mathbb{R}^4, \quad \text{Nil}^3 \times \mathbb{R}, \quad \text{Nil}^4, \quad \text{Sol}^3 \times \mathbb{R}.$$

These Lie groups are all diffeomorphic to  $\mathbb{R}^4$ . The Lie groups  $\text{Nil}^3$ ,  $\text{Nil}^4$  are nilpotent and  $\text{Sol}^3$  is a particular solvable (non nilpotent) Lie group.

In particular, if the total space  $M^4$  of an orientable  $T^2$ -bundle over a  $\mathbb{T}^2$  is a solvmanifold, then it must be the compact quotient of one of the above Lie groups  $G$ . It is well known that all the orientable  $T^2$ -bundles over  $\mathbb{T}^2$  admit symplectic structures (see [4]). The notion of invariant almost-Kähler structure makes sense for orientable  $T^2$ -bundles over  $\mathbb{T}^2$ , meaning one induced from a left-invariant structure on  $G$  which is invariant by the discrete subgroup  $\Gamma$ .

As a main result of [2] it was shown that if  $M^4 = \Gamma \backslash G$  is an orientable  $T^2$ -bundle over a  $\mathbb{T}^2$  with  $G = \text{Nil}^3 \times \mathbb{R}$  or  $\text{Nil}^4$ , and if  $M^4$  admits an invariant Lagrangian almost-Kähler structure  $(\Omega, J)$ , then for every normalized volume form  $\sigma = e^F \Omega^2$  with  $F \in C^\infty(\mathbb{T}^2)$ , the corresponding Calabi-Yau problem has a unique solution.

The Lagrangian condition may or may not apply in the case of  $G = \text{Nil}^3 \times \mathbb{R}$ , but is automatic when  $M^4$  is modelled on the 3-step nilpotent Lie group  $\text{Nil}^4$ . In the case of  $G = \text{Sol}^3$  every invariant almost-Kähler on  $\Gamma \backslash G$  is non-Lagrangian.

The aim of this paper is to extend the main result in [2] to the *non-Lagrangian* cases, i.e. to some  $T^2$ -fibrations modelled on  $\text{Nil}^3 \times \mathbb{R}$  and to all the  $T^2$ -fibrations modelled on  $\text{Sol}^3 \times \mathbb{R}$ .

Our main result is the following

**Theorem 1.1.** *Let  $M^4 = \Gamma \backslash G$  be an orientable  $T^2$ -bundle over a  $\mathbb{T}^2$  with  $G = \text{Nil}^3 \times \mathbb{R}$  or  $\text{Sol}^3 \times \mathbb{R}$ , and suppose that  $M^4$  admits an invariant non-Lagrangian almost-Kähler*

structure  $(\Omega, J)$ . Then for every normalized volume form  $\sigma = e^F \Omega^2$  with  $F \in C^\infty(\mathbb{T}^2)$ , the corresponding Calabi–Yau problem has a unique solution.

The proof of this theorem consists in showing that the Calabi–Yau problem can be reduced to a single elliptic Monge–Ampère equation which has solution.

The trick of reducing the problem to a Monge–Ampère equation was the core of [2], but the class of equations which appear in the present paper differs from the ones considered in [2].

As a consequence we show that for every total space  $M^4$  of an orientable  $T^2$ -bundle over a  $\mathbb{T}^2$  endowed with an invariant almost-Kähler structure  $(\Omega, J)$  the Calabi–Yau problem has a solution for every normalized  $T^2$ -invariant volume form.

The paper is organized as follows: In Section 2 we recall the classification of  $T^2$ -bundles over  $\mathbb{T}^2$  and we briefly describe the main result in [2]. Sections 3 and 4 contain the proof of Theorem 1.1 where the case of  $G = \text{Nil}^3 \times \mathbb{R}$  and  $G = \text{Sol}^3 \times \mathbb{R}$  are treated separately. In each of the two cases we can reduce the problem to a Monge–Ampère equation for which we show the existence of a solution.

## 2. THE CALABI–YAU EQUATION ON $T^2$ -BUNDLES OVER $\mathbb{T}^2$

Orientable  $T^2$ -bundles over a  $\mathbb{T}^2$  were classified by Fukuhara and Sakamoto in [3] and it was shown by Ue in [10, 11] that all these manifolds are infra-solvmanifolds. A compact manifold  $M$  is called an *infra-solvmanifold* if it admits a finite cover  $\pi: \tilde{M} \rightarrow M$ , where  $\tilde{M} = \tilde{\Gamma} \backslash G$  is the compact quotient of a solvable Lie group  $G$  by a lattice  $\tilde{\Gamma}$ . Alternatively,  $M$  can be written as a quotient  $M = \Gamma \backslash G$ , where  $\Gamma$  is a discrete group containing a lattice  $\tilde{\Gamma}$  of  $G$  such that  $\tilde{\Gamma} \backslash \Gamma$  is finite. In the case that  $\tilde{\Gamma}$  is a lattice,  $M$  is simply called a *solvmanifold*.

It turns out that in the classification of  $T^2$ -bundles over  $\mathbb{T}^2$ , the solvable Lie group  $G$  must be one of the following four patterns

$$(2.1) \quad \mathbb{R}^4, \quad \text{Nil}^3 \times \mathbb{R}, \quad \text{Nil}^4, \quad \text{Sol}^3 \times \mathbb{R},$$

while the classification of the possible  $\Gamma$ 's determines eight families. For the Lie groups  $\text{Nil}^3 \times \mathbb{R}, \text{Nil}^4, \text{Sol}^3 \times \mathbb{R}$  we have the following description:

- (1)  $\text{Nil}^3$  is the 3-dimensional Heisenberg group of matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

and  $\text{Nil}^3 \times \mathbb{R}$  is a 2-step nilpotent Lie group which can be regarded as  $\mathbb{R}^4$  with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + x, y_0 + y, z_0 + z + x_0 y, t_0 + t).$$

(2)  $\text{Nil}^4 = \mathbb{R} \ltimes \mathbb{R}^3$  is the 3-step nilpotent Lie group of real matrices

$$\begin{pmatrix} 1 & t & \frac{1}{2}t^2 & x \\ 0 & 1 & t & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3)  $\text{Sol}^3 = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^2$  is a unimodular 2-step solvable Lie group with  $\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

and  $\text{Sol}^3 \times \mathbb{R}$  can be regarded as  $\mathbb{R}^4$  with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + e^{t_0}x, y_0 + e^{-t_0}y, z_0 + z, t_0 + t).$$

The diffeomorphism classes of (the total space of)  $T^2$ -bundles over  $\mathbb{T}^2$  can be summarized in Geiges' eight families [4, Table 1], which can be explicitly described in terms of the generators of the discrete groups  $\Gamma$ , the monodromy matrices along the two curves generating  $\pi_1(\mathbb{T}^2)$ , as well as the Euler class for the corresponding  $T^2$ -bundle.

In the case of  $G = \text{Nil}^3 \times \mathbb{R}$  one has two inequivalent fibrations

$$\begin{aligned} \pi_{xy} &: M^4 \rightarrow \mathbb{T}_{xy}^2, \\ \pi_{yt} &: M^4 \rightarrow \mathbb{T}_{yt}^2 \end{aligned}$$

induced from the following coordinate mappings:

$$\begin{aligned} (x, y, z, t) &\mapsto (x, y), \\ (x, y, z, t) &\mapsto (y, t). \end{aligned}$$

If  $\Gamma$  is not a lattice of  $G$ , we have that  $\Gamma$  contains a lattice  $\tilde{\Gamma}$  of  $G$  such that the quotient  $\tilde{\Gamma} \backslash \Gamma$  is a finite group. Therefore there exists a covering map  $p: \tilde{\Gamma} \backslash G \rightarrow \Gamma \backslash G$  which preserves the  $T^2$ -bundle structure over  $\mathbb{T}^2$ .

We recall that an *almost-Kähler structure* on a manifold  $M$  is a pair  $(\Omega, J)$ , where  $\Omega$  is a symplectic form and  $J$  is a endomorphism of the tangent bundle to  $M$  satisfying  $J^2 = -I$  and

$$\Omega(JX, JY) = \Omega(X, Y), \quad \Omega(Z, JZ) > 0$$

for every tangent vector fields  $X, Y, Z$  on  $M$  with  $Z$  nowhere vanishing. Every almost-Kähler structure induces the Riemannian metric

$$g(X, Y) = \Omega(X, JY).$$

In this paper (as in [2]) we consider on the total space  $M^4 = \Gamma \backslash G$  of  $T^2$ -bundles over  $\mathbb{T}^2$  invariant almost-Kähler structures, i.e. ones induced from left-invariant structures on  $G$  which are invariant by the discrete group  $\Gamma$  and we study for these almost-Kähler manifolds the Calabi-Yau problem. In particular every invariant almost-Kähler structure on  $M^4 = \Gamma \backslash G$  induces an invariant almost-Kähler structure on the solvmanifold  $\tilde{\Gamma} \backslash G$ .

The case  $G = \mathbb{R}^4$  (which corresponds to two of the Geiges' families) is not interesting from our point of view, since in this case every invariant almost-Kähler structure is Kähler and Yau's theorem can be applied. For the other cases we have to distinguish the Lagrangian case from the non-Lagrangian one. If  $M^4$  is modelled on  $G = \text{Nil}^4$  or on  $G = \text{Nil}^3 \times \mathbb{R}$  with bundle structure given by the projection  $\pi_{xy}$ , then every invariant almost-Kähler is Lagrangian. If  $M^4$  is modelled on  $\text{Nil}^3 \times \mathbb{R}$  with bundle structure given by the projection  $\pi_{yt}$  then it admits Lagrangian and non-Lagrangian almost-Kähler structures as well. In the case  $G = \text{Sol}^3 \times \mathbb{R}$ , every invariant almost-Kähler structure is non-Lagrangian.

Let now  $M^4 = \Gamma/G$  be an orientable  $T^2$ -bundle over  $\mathbb{T}^2$  and denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . Then every basis  $(e^i)$  of the dual space  $\mathfrak{g}^*$  induces a global frame of 1-forms on  $M^4$ . Furthermore we fix an invariant almost-Kähler structure  $(\Omega, J)$  on  $M^4$ . Let  $\sigma = e^F \Omega^2$  be a volume form and let  $F$  be a smooth map on the base  $\mathbb{T}^2$  of  $M^4$  satisfying

$$\int_{\mathbb{T}^2} (e^F - 1) = 0.$$

Then in this case the Calabi-Yau problem reads as

$$(2.2) \quad \begin{cases} (\Omega + da)^2 = e^F \Omega^2, \\ J(da) = da, \end{cases}$$

on  $M^4$  whose components with respect to the coframe  $(e^i)$  are defined on the torus  $\mathbb{T}^2$ . Thus the Calabi-Yau problem reduces to a system of partial differential equations on the base  $\mathbb{T}^2$ .

Although the system (2.2) depends on the choice of  $G$ ,  $(\Omega, J)$  and the structure of  $T^2$ -fibration, for all the cases we can proceed in the following way: first we parametrize  $(\Omega, J)$  using a suitable invariant coframe on  $M^4$  in order to simplify the formulation of (2.2) as far as possible and then we perform a suitable change of variables transforming the system (2.2) in a Monge-Ampère equation on the base  $\mathbb{T}^2$ .

The Lagrangian cases have been considered in [2], where it has been proved that has a unique solution. In the next two sections we will consider the non-Lagrangian cases for the manifolds modelled on  $\text{Nil}^3 \times \mathbb{R}$  and  $\text{Sol}^3 \times \mathbb{R}$ .

### 3. MANIFOLDS MODELLED ON $\text{Nil}^3 \times \mathbb{R}$ : THE NON-LAGRANGIAN CASE

In this section we study the Calabi-Yau problem for  $T^2$ -bundles over  $\mathbb{T}^2$  modelled on  $\text{Nil}^3 \times \mathbb{R}$  and equipped with an invariant non-Lagrangian almost-Kähler structure.

The structure of  $T^2$ -bundle over a  $\mathbb{T}^2$  is then induced by the projection  $\pi_{yt}$  onto the torus  $\mathbb{T}_{yt}^2$ . The total space  $M^4$  of the  $T^2$ -fibration has the global invariant coframe

$$e^1 = dy, \quad e^2 = dx, \quad e^3 = dt, \quad e^4 = dz - xdy$$

which satisfies the structure equations

$$(3.1) \quad de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}.$$

**Lemma 3.1.** *Let  $M^4$  be the total space of an oriented  $T^2$ -bundle over  $\mathbb{T}^2$  modelled on  $\text{Nil}^3 \times \mathbb{R}$  and induced by the projection  $\pi_{yt}$  onto the torus  $\mathbb{T}_{yt}^2$ .*

*Let  $(\Omega, J)$  be an invariant almost-Kähler structure on  $M^4$  with induced Riemannian metric  $g$ . Then there exists an orthonormal invariant coframe  $(f^i)$  for which*

$$(3.2) \quad \Omega = f^{14} + f^{23},$$

and

$$(3.3) \quad f^1 \in \langle e^1 \rangle, \quad g(e^3, f^2) = 0, \quad g(e^3, f^3)g(e^3, f^4) \geq 0.$$

*Proof.* We can certainly find an orthonormal invariant coframe  $(f^i)$  for which (3.2) is valid and  $f^1 \in \langle e^1 \rangle$ . Since  $f^4 = J(f^1)$ , we still have the freedom of rotate  $f^{23}$  in the plane orthogonal to  $\langle f^1, f^4 \rangle$ . After a suitable rotation we obtain  $g(e^3, f^2) = 0$ . Eventually, we may invert the direction of  $f^2$  and  $f^3$  to meet the condition  $g(e^3, f^3)g(e^3, f^4) \geq 0$ , without reversing  $f^{23}$ .  $\square$

**3.1. The Calabi-Yau equation on  $M^4$ .** Consider on  $M^4$  an invariant non-Lagrangian almost-Kähler structure  $(\Omega, J)$  with induced Riemannian metric  $g$ . Let  $\sigma = e^F \Omega^2$  be a volume form where  $F = F(y, t)$  is a smooth map on the base satisfying

$$(3.4) \quad \int_{\mathbb{T}^2} (e^F - 1) = 0.$$

Consider the Calabi-Yau equation

$$\begin{cases} (\Omega + da)^2 = \sigma, \\ J(da) = da, \end{cases}$$

where  $a$  is a 1-form on  $M^4$  whose components with respect to the basis  $(e^i)$  depend on  $(y, t)$  only. Let  $(f^i)$  be a coframe as in Lemma 3.1 and set

$$G_j^i = g(e^i, f^j);$$

then we have

$$e^i = G_j^i f^j.$$

In particular we have

$$(3.5) \quad e^1 = G_1^1 f^1$$

and

$$(3.6) \quad G_1^1 \neq 0, \quad G_2^1 = G_3^1 = G_4^1 = 0.$$

Let  $H = G^{-1}$  be the inverse matrix of  $G = (G_j^i)$ . Then

$$f^i = H_j^i e^j.$$

From (3.5) and (3.6), we have

$$H_1^1 = (G_1^1)^{-1} \neq 0,$$

and

$$(3.7) \quad H_2^1 = H_3^1 = H_4^1 = 0.$$

Thanks to structure equations (3.1), we have

$$df^i = H_j^i de^j = H_4^i de^4 = H_4^i e^{12} = H_4^i G_1^1 (G_2^2 f^{12} + G_3^2 f^{13} + G_4^2 f^{14}).$$

The condition that  $(\Omega, J)$  is non-Lagrangian implies that

$$G_4^3 \neq 0.$$

Moreover, since  $G_2^3 = g(e^3, f^2) = 0$ , we have

$$(3.8) \quad e^3 = G_1^3 f^1 + G_3^3 f^3 + G_4^3 f^4,$$

where

$$(3.9) \quad G_3^3 G_4^3 \geq 0,$$

thanks to (3.3).

Differentiating (3.8) we get

$$G_3^3 df^3 + G_4^3 df^4 = (G_3^3 H_4^3 + G_4^3 H_4^4) e^{12} = 0,$$

i.e.

$$(3.10) \quad G_3^3 H_4^3 + G_4^3 H_4^4 = 0.$$

Furthermore, the symplectic condition  $d\Omega = 0$  gives

$$df^{23} = 0$$

that is

$$\begin{aligned} 0 &= H_4^2 G_1^1 (G_2^2 f^{12} + G_3^2 f^{13} + G_4^2 f^{14}) \wedge f^3 - H_4^3 G_1^1 f^2 \wedge (G_2^2 f^{12} + G_3^2 f^{13} + G_4^2 f^{14}) \\ &= G_1^1 (H_4^2 G_2^2 + G_1^1 H_4^3 G_3^2) f^{123} - G_1^1 H_4^2 G_4^2 f^{134} + G_1^1 H_4^3 G_4^2 f^{124}. \end{aligned}$$

It follows that

$$(3.11) \quad H_4^2 G_2^2 + H_4^3 G_3^2 = 0, \quad H_4^2 G_4^2 = H_4^3 G_4^2 = 0.$$

From (3.10) and (3.11) we have

$$(3.12) \quad G_4^2 G_4^3 H_4^4 = G_4^2 (G_3^3 H_4^3 + G_4^3 H_4^4) = 0,$$

hence, since  $H_4^1 = 0$ , and  $H_4^2$ ,  $H_4^3$  and  $H_4^4$  cannot vanish all together, from (3.7), (3.11) and (3.12) we obtain that

$$G_4^2 = 0.$$

Write

$$a = a_k f^k$$

and compute

$$da = (G_1^1 a_{2,y} + G_1^1 G_2^2 H_4^k a_k + G_1^3 a_{2,t}) f^{12} + (G_1^1 a_{3,y} + G_1^1 G_3^2 H_4^k a_k + G_1^3 a_{3,t} - G_3^3 a_{1,t}) f^{13} \\ - G_4^3 a_{2,t} f^{24} + (G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t}) f^{14} - G_3^3 a_{2,t} f^{23} + (G_3^3 a_{4,t} - G_4^3 a_{3,t}) f^{34}.$$

Hence  $da$  is of type  $(1, 1)$  with respect to  $J$  if and only if

$$\begin{cases} G_1^1 a_{2,y} + G_1^1 G_2^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{2,t} = -G_3^3 a_{4,t} + G_4^3 a_{3,t}, \\ G_1^1 a_{3,y} + G_1^1 G_3^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{3,t} - G_3^3 a_{1,t} = -G_3^3 a_{2,t} \end{cases}$$

and in this case  $da$  reduces to

$$da = (-G_3^3 a_{4,t} + G_4^3 a_{3,t}) f^{12} - G_3^3 a_{2,t} f^{13} - G_4^3 a_{2,t} f^{24} \\ + (G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t}) f^{14} - G_3^3 a_{2,t} f^{23} + (G_3^3 a_{4,t} - G_4^3 a_{3,t}) f^{34}.$$

The Calabi-Yau equation now reads as

$$e^F = (1 + G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t})(1 - G_3^3 a_{2,t}) - G_3^3 G_4^3 (a_{2,t})^2 - (-G_3^3 a_{4,t} + G_4^3 a_{3,t})^2$$

and the Calabi-Yau problem is equivalent to the following system of partial differential equations:

$$(3.13) \quad \begin{cases} G_1^1 a_{2,y} + G_1^1 G_2^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{2,t} + G_3^3 a_{4,t} - G_4^3 a_{3,t} = 0, \\ G_1^1 a_{3,y} + G_1^1 G_3^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{3,t} - G_3^3 a_{1,t} + G_3^3 a_{2,t} = 0, \\ (1 + G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t})(1 - G_3^3 a_{2,t}) - \\ \quad - G_3^3 G_4^3 (a_{2,t})^2 - (-G_3^3 a_{4,t} + G_4^3 a_{3,t})^2 = e^F. \end{cases}$$

In the system (3.13) the parameter  $G_3^3$  has a special role. We will study separately the cases  $G_3^3 = 0$  and  $G_3^3 \neq 0$ .

**3.2. The case  $G_3^3 = 0$ .** This case is quite trivial since condition  $G_3^3 = 0$  implies  $dt \in \langle f^1, f^4 \rangle$  and  $f^{14} = dy \wedge dt$ . Therefore if  $G_3^3 = 0$  the corresponding Calabi-Yau equation has the explicit solution

$$\tilde{\Omega} = (e^F - 1) f^{14} + f^{23}.$$

**3.3. The Case  $G_3^3 \neq 0$ .** Under this assumption we consider the transformation

$$\begin{aligned} a_1 &= -G_3^3 u_t - G_1^1 (H_4^2 G_3^2 + H_4^4 G_2^2) u, \\ a_2 &= -G_3^3 u_t - H_4^4 G_1^1 G_2^2 u, \\ a_3 &= -H_4^4 G_1^1 G_3^2 u, \\ a_4 &= G_1^1 u_y + G_1^3 u_t. \end{aligned}$$

A long but straightforward computation shows that the first two equations of system (3.13) are identically satisfied, while the third one becomes

$$(3.14) \quad (u_{yy} + B_{11} u_t + C_{11})(u_{tt} + B_{22} u_t + C_{22}) - (u_{yt} + B_{12} u_t + C_{12})^2 = E_1 + E_2 e^F,$$



where

$$\begin{aligned}
B_{11} &= \frac{G_2^2(G_1^3)^2 H_4^4}{G_1^1 G_3^3} - 2 \frac{G_3^2 G_1^3 G_4^3 H_4^4}{G_1^1 G_3^3} + \frac{G_3^2 G_4^3 H_4^2}{G_1^1}, \\
B_{12} &= -\frac{G_2^2 G_1^3 H_4^4}{G_3^3} + \frac{G_3^2 G_4^3 H_4^4}{G_3^3}, \quad B_{22} = \frac{G_1^1 G_2^2 H_4^4}{G_3^3}, \\
C_{11} &= \frac{1}{(G_1^1)^2} + \frac{(G_1^3)^2}{(G_1^1)^2 (G_3^3)^2} + \frac{G_4^3}{(G_1^1)^2 G_3^3}, \\
C_{12} &= -\frac{G_1^3}{G_1^1 (G_3^3)^2}, \quad C_{22} = \frac{1}{(G_3^3)^2}, \\
E_1 &= \frac{G_3^3 G_4^3}{(G_1^1)^2 (G_3^3)^4}, \quad E_2 = (G_1^1 G_3^3)^2.
\end{aligned}$$

In particular we have

$$B_{11} B_{22} - (B_{12})^2 = 0$$

and

$$C_{11} C_{22} - (C_{12})^2 = E_1 + E_2.$$

#### 4. MANIFOLDS MODELLED ON $\text{Sol}^3 \times \mathbb{R}$

In this section we study the Calabi-Yau equation for the total spaces  $M^4$  of  $T^2$ -bundles over torus  $\mathbb{T}^2$  modelled on  $\text{Sol}^3 \times \mathbb{R}$ .

Since the Lie group  $\text{Sol}^3 \times \mathbb{R}$  can be seen as  $\mathbb{R}^4$  equipped with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + x, y_0 + y, z_0 + e^x z, t_0 + e^{-x} t)$$

$M^4$  inherits the global invariant coframe

$$(4.1) \quad e^1 = dx, \quad e^2 = dy, \quad e^3 = e^x dz, \quad e^4 = e^{-x} dt$$

satisfying the following structure equations

$$(4.2) \quad de^1 = de^2 = 0, \quad de^3 = e^{13}, \quad de^4 = -e^{14}.$$

Moreover, invariant almost-Kähler structure on  $M^4$  should be parametrized as claimed in the following lemma proved in [2]

**Lemma 4.1.** *Let  $(\Omega, J)$  be an invariant almost-Kähler structure on the total space  $M^4$  of a  $T^2$ -bundle over a torus  $\mathbb{T}^2$  modelled on  $\text{Sol}^3 \times \mathbb{R}$ . Let  $g$  be the induced Riemannian metric. Then there exists an orthonormal global coframe  $(f^i)$  for which*

$$\Omega = f^{12} + f^{34},$$

and

$$(4.3) \quad f^1 \in \langle e^1 \rangle, \quad f^3 \in \langle e^3 \rangle, \quad f^4 \in \langle e^3, e^4 \rangle,$$

with

$$(4.4) \quad g(e^1, f^1) > 0.$$

Notice that in this case every invariant almost-Kähler structure is non-Lagrangian.

**4.1. The Calabi-Yau equation on  $M^4$ .** Let  $(\Omega, J)$  be an invariant almost-Kähler structure on  $M^4$ ,  $(f^k)$  be a coframe as in the previous lemma and  $\sigma = e^F \Omega^2$  be a volume form where  $F = F(x, y) \in C^\infty(\mathbb{T}^2)$  satisfies the condition

$$\int_{\mathbb{T}^2} (e^F - 1) = 0.$$

Then we consider the Calabi-Yau equation

$$(4.5) \quad (\Omega + da)^2 = \sigma,$$

where

$$a = \sum_{k=1}^4 a_k f^k,$$

is a 1-form whose components  $a_k$  are functions on the base  $\mathbb{T}^2$ .

Let  $g$  be the Riemannian metric induced by  $(\Omega, J)$  and set

$$G_j^i = g(e^i, f^j).$$

Then

$$(4.6) \quad e^i = G_j^i f^j$$

and

$$f^i = H_j^i e^j,$$

where  $H = G^{-1}$  is the inverse matrix of  $G = (G_j^i)$ . In particular this implies that

$$G_4^3 = H_4^2 G_2^2 + H_4^4 G_4^2 = H_3^3 G_3^4 + H_3^4 G_4^4 = 0.$$

From (4.3) we have that

$$f^1 = H_1^1 e^1, \quad f^2 = H_2^2 e^2 + H_3^2 e^3 + H_4^2 e^4, \quad f^3 = H_3^3 e^3, \quad f^4 = H_3^4 e^3 + H_4^4 e^4.$$

Then, making use of (4.2) and (4.6), we obtain

$$\begin{aligned} df^1 &= 0, & df^2 &= G_1^1 (H_3^2 G_3^3 - H_4^2 G_3^4) f^{13} - G_1^1 H_4^2 G_4^4 f^{14}, \\ df^3 &= G_1^1 f^{13}, & df^4 &= G_1^1 (H_3^4 G_3^3 - H_4^4 G_3^4) f^{13} - G_1^1 f^{14}. \end{aligned}$$

Let

$$a = a_1 f^1 + a_2 f^2 + a_3 f^3 + a_4 f^4$$

be a  $T^2$ -invariant form on  $M^4$ .

We have

$$\begin{aligned}
da = & (G_1^1 a_{2,x} - G_2^2 a_{1,y}) f^{12} \\
& + \left( G_1^1 a_{3,x} - G_3^2 a_{1,y} + G_1^1 (H_3^2 G_3^3 - H_4^2 G_3^4) a_2 + G_1^1 a_3 + G_1^1 (H_3^4 G_3^3 - H_4^4 G_3^4) a_4 \right) f^{13} \\
& + \left( G_1^1 a_{4,x} - G_4^2 a_{1,y} - G_1^1 H_4^2 G_4^4 a_2 - G_1^1 a_4 \right) f^{14} \\
& + (G_2^2 a_{3,y} - G_3^2 a_{2,y}) f^{23} + (G_2^2 a_{4,y} - G_4^2 a_{2,y}) f^{24} + (G_3^2 a_{4,y} - G_4^2 a_{3,y}) f^{34}.
\end{aligned}$$

Hence  $da$  is  $J$ -invariant if and only if its components satisfy

$$\begin{cases} G_1^1 a_{3,x} - G_3^2 a_{1,y} + G_1^1 (H_3^2 G_3^3 - H_4^2 G_3^4) a_2 + G_1^1 a_3 + G_1^1 (H_3^4 G_3^3 - H_4^4 G_3^4) a_4 \\ \quad \quad \quad = G_2^2 a_{4,y} - G_4^2 a_{2,y}, \\ G_1^1 a_{4,x} - G_4^2 a_{1,y} - G_1^1 H_4^2 G_4^4 a_2 - G_1^1 a_4 = G_3^2 a_{2,y} - G_2^2 a_{3,y}, \end{cases}$$

and equation (4.5) becomes

$$\begin{aligned}
(1 + G_1^1 a_{2,x} - G_2^2 a_{1,y})(1 + G_3^2 a_{4,y} - G_4^2 a_{3,y}) \\
- (G_2^2 a_{4,y} - G_4^2 a_{2,y})^2 - (G_2^2 a_{3,y} - G_3^2 a_{2,y})^2 = e^F.
\end{aligned}$$

Therefore the Calabi-Yau problem reduces to the following system of partial differential equations

$$(4.7) \quad \begin{cases} G_1^1 a_{3,x} - G_3^2 a_{1,y} + G_1^1 (H_3^2 G_3^3 - H_4^2 G_3^4) a_2 + G_1^1 a_3 + G_1^1 (H_3^4 G_3^3 - H_4^4 G_3^4) a_4 \\ \quad \quad \quad = G_2^2 a_{4,y} - G_4^2 a_{2,y}, \\ G_1^1 a_{4,x} - G_4^2 a_{1,y} - G_1^1 H_4^2 G_4^4 a_2 - G_1^1 a_4 = G_3^2 a_{2,y} - G_2^2 a_{3,y}, \\ (1 + G_1^1 a_{2,x} - G_2^2 a_{1,y})(1 + G_3^2 a_{4,y} - G_4^2 a_{3,y}) \\ \quad \quad \quad - (G_2^2 a_{4,y} - G_4^2 a_{2,y})^2 - (G_2^2 a_{3,y} - G_3^2 a_{2,y})^2 = e^F. \end{cases}$$

**4.2. Reduction of (4.7) to a single equation.** Consider  $u \in C^2(\mathbb{T}^2)$  such that

$$(4.8) \quad \int_{\mathbb{T}^2} u = 0,$$

and let

$$\left\{ \begin{aligned} a_1(x, y) &= -\frac{H_1^1 G_2^2 (u_y(x, y) - u_y(x, 0)) + 2H_4^4 G_3^4 (u(x, y) - u(x, 0))}{(G_3^2)^2 + (G_4^2)^2} \\ &\quad - \frac{G_1^1 H_2^2}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^y (u_{xx}(x, t) - u(x, t)) dt - y \int_0^1 (u_{xx}(x, t) - u(x, t)) dt \right), \\ a_2(x, y) &= -\frac{1}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x \left( \int_0^1 u(s, t) dt \right) ds - \int_0^1 (u_x(x, t) - u_x(0, t)) dt \right), \\ a_3(x, y) &= -\frac{H_2^2 G_3^2 u_x(x, y) + H_1^1 G_4^2 u_y(x, y) - H_2^2 (G_3^2 - 2G_4^2 H_4^4 G_3^4) u(x, y)}{(G_3^2)^2 + (G_4^2)^2} \\ &\quad - \frac{H_2^2 G_3^2}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x \left( \int_0^1 u(s, t) dt \right) ds - \int_0^1 (u_x(x, t) - u_x(0, t)) dt \right), \\ a_4(x, y) &= -\frac{H_2^2 G_4^2 u_x(x, y) - H_1^1 G_3^2 u_y(x, y) + H_2^2 G_4^2 u(x, y)}{(G_3^2)^2 + (G_4^2)^2} \\ &\quad - \frac{H_2^2 G_4^2}{(G_3^2)^2 + (G_4^2)^2} \left( \int_0^x \left( \int_0^1 u(s, t) dt \right) ds - \int_0^1 (u_x(x, t) - u_x(0, t)) dt \right). \end{aligned} \right.$$

Thanks to condition (4.8) we have that the functions  $a_1$  to  $a_4$  are periodic. A long computation shows that the first two equations of system (4.7) are identically satisfied, while the third one becomes:

$$(4.9) \quad (u_{xx} + B_{11}u_y + C_{11} + Du)(u_{yy} + B_{22}u_y + C_{22}) - (u_{xy} + B_{12}u_y)^2 = E_1 + E_2 e^F$$

where

$$\begin{aligned} B_{11} &= \frac{2 H_1^1 G_2^2 G_3^2 (G_4^2 + G_3^2 H_4^4 G_3^4)}{(G_3^2)^2 + (G_4^2)^2}, \\ B_{12} &= \frac{(G_4^2)^2 - (G_3^2)^2 + 2G_3^2 G_4^2 H_4^4 G_3^4}{(G_3^2)^2 + (G_4^2)^2}, \\ B_{22} &= -\frac{2G_1^1 H_2^2 G_4^2 (G_3^2 - G_4^2 H_4^4 G_3^4)}{(G_3^2)^2 + (G_4^2)^2}, \\ C_{11} &= H_1^1 \left( (G_2^2)^2 + (G_3^2)^2 + (G_4^2)^2 \right), \quad C_{22} = G_1^1, \\ D &= -1, \quad E_1 = (G_2^2)^2, \quad E_2 = (G_3^2)^2 + (G_4^2)^2. \end{aligned}$$

In particular we have

$$B_{11}B_{22} - (B_{12})^2 = -1$$

and

$$C_{11}C_{22} - (C_{12})^2 = E_1 + E_2.$$

## 5. THE MONGE-AMPÈRE EQUATION

Both equations (3.14) and (4.9) are generalized Monge-Ampère equations of the following type:

$$(5.1) \quad A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + E_2 e^F,$$

where

$$\begin{aligned} A_{11}[u] &= u_{xx} + B_{11}u_y + C_{11} + Du, \\ A_{12}[u] &= u_{xy} + B_{12}u_y + C_{12}, \\ A_{22}[u] &= u_{yy} + B_{22}u_y + C_{22}, \end{aligned}$$

with  $B_{ij}$ ,  $C_{ij}$ ,  $D$ ,  $E_i$  real numbers such that

$$(5.2) \quad C_{11} + C_{22} > 0, \quad D \leq 0,$$

$$(5.3) \quad E_1 > 0, \quad E_2 > 0,$$

$$(5.4) \quad B_{11}B_{22} - (B_{12})^2 = D$$

and

$$(5.5) \quad C_{11}C_{22} - (C_{12})^2 = E_1 + E_2.$$

Moreover

$$(5.6) \quad F \in C^\infty(\mathbb{T}^2)$$

and satisfies the condition

$$(5.7) \quad \int_{\mathbb{T}^2} (e^F - 1) = 0.$$

In Theorem 5.7 we shall prove that equation (5.1) has a solution belonging to  $C^\infty(\mathbb{T}^2)$  and satisfying the condition

$$(5.8) \quad \int_{\mathbb{T}^2} u = 0.$$

For all  $n \in \mathbb{N}$ ,  $0 < \epsilon < 1$ , consider the semi-norms

$$\begin{aligned} |u|_{C^n} &= \max_{0 \leq j \leq n} \sup_{(x,y) \in \mathbb{R}^2} |\partial_x^j \partial_y^{n-j} u(x, y)| \\ |u|_{C^{n,\epsilon}} &= \max_{0 \leq j \leq n} \sup_{(x,y) \in \mathbb{R}^2} \sup_{(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{|\partial_x^j \partial_y^{n-j} u(x+h, y+k) - \partial_x^j \partial_y^{n-j} u(x, y)|}{(h^2 + k^2)^{\epsilon/2}} \end{aligned}$$

and the norms

$$\|u\|_{C^n} = \max_{0 \leq k \leq n} |u|_{C^k}, \quad \|u\|_{C^{n,\epsilon}} = \max\{\|u\|_{C^n}, |u|_{C^{n,\epsilon}}\}.$$

**Lemma 5.1.** *Under the hypotheses (5.3), for all  $u \in C^2(\mathbb{T}^2)$  satisfying (5.1) we have that*

$$(5.9) \quad \begin{cases} A_{11}[u] > 0, \\ A_{22}[u] > 0. \end{cases}$$

*Proof.* Equation (5.1) implies that  $A_{11}[u]A_{22}[u] > E_1$ . Then  $A_{11}[u]$  and  $A_{22}[u]$  never vanish and have the same sign. At a point where  $u$  reaches its minimum value, we have  $u_y = 0$  and  $u_{yy} \leq 0$ . Then

$$A_{22}[u] = u_{yy} + C_{22} > 0$$

and both  $A_{11}[u]$  and  $A_{22}[u]$  must be positive everywhere.  $\square$

**Lemma 5.2.** *Consider a function  $u \in C^2(\mathbb{T}^2)$  satisfying equation (5.1). Under the hypotheses (5.2) and (5.3) we have that*

$$(5.10) \quad Du(x, y) \geq C_{11}, \quad \forall (x, y) \in \mathbb{R}^2.$$

*Proof.* Consider a point where  $Du$  attains its minimum value. Since  $D \leq 0$ , this corresponds to a point where  $u$  reaches its maximum value. Then we have  $u_y = 0$  and  $u_{xx} \leq 0$  and from (5.9) we have

$$C_{11} + Du \geq u_{xx} + C_{11} + Du > 0,$$

which implies

$$Du \geq C_{11},$$

at the maximum and therefore everywhere.  $\square$

We need Lemma 6.3 of [2]:

**Lemma 5.3.** *Consider  $w \in C^2(\mathbb{T})$  and two real numbers  $\alpha$  and  $\beta$  such that*

$$(5.11) \quad w''(t) + \alpha w'(t) \geq \beta, \quad \forall t \in \mathbb{R}.$$

*Then we have*

$$(5.12) \quad |w'(t)| \leq 2|\beta|e^{2|\alpha|}, \quad \forall t \in \mathbb{R}.$$

**Theorem 5.4.** *Assume hypotheses (5.2) and (5.5) are satisfied. Then all solutions of (5.1) satisfy the following estimate:*

$$\|u\|_{C^2} \leq 2(|B_{11}| + 1)|B_{22}|e^{2C_{22}} + C_{11} + C_{22}.$$

*Proof.* From (5.9) we obtain that

$$u_{yy} + B_{22}u_y \geq -C_{22},$$

hence from Lemma 5.3 we obtain that

$$(5.13) \quad |u_y| \leq 2|B_{22}|e^{2C_{22}}.$$

From (5.9), (5.10) and (5.13), we obtain

$$u_{xx} \geq -2|B_{11}B_{22}|e^{2C_{22}} - C_{11} - C_{22},$$

hence from Lemma 5.3 we obtain

$$(5.14) \quad |u_x| \leq 2 |B_{11} B_{22}| e^{2C_{22}} + C_{11} + C_{22}.$$

Now consider a point  $(x_0, y_0) \in [0, 1] \times [0, 1]$  where  $u$  vanishes. Then we have

$$\begin{aligned} u(x, y) &= \int_0^1 u_x((1-t)x + tx_0, (1-t)y + ty_0) dt (x - x_0) \\ &\quad + \int_0^1 u_y((1-t)x + tx_0, (1-t)y + ty_0) dt (y - y_0), \end{aligned}$$

which, together periodicity, implies

$$|u|_{C^0} \leq 2 |u|_{C^1}.$$

This estimate, together (5.13) and (5.14), implies

$$|u|_{C^0} \leq 2(|B_{11}| + 1) |B_{22}| e^{2C_{22}} + C_{11} + C_{22}. \quad \square$$

Let  $\tau \in [0, 1]$  and set

$$(5.15) \quad \mathfrak{S}_\tau = \left\{ u \in C^2(\mathbb{T}^2) \mid A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-\tau)E_2 + \tau E_2 e^F, \int_{\mathbb{T}^2} u = 0 \right\}.$$

**Theorem 5.5.** *Assume hypotheses (5.2) to (5.6) are satisfied. Then*

$$\mathfrak{S}_\tau \subset C^{2,1/2}(\mathbb{T}^2), \quad \forall \tau \in [0, 1],$$

and

$$\sup_{0 \leq \tau \leq 1} \sup_{u \in \mathfrak{S}_\tau} \|u\|_{C^{2,1/2}} < \infty.$$

*Proof.* Thanks to lemma 5.1 and hypothesis (5.3) the equation

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-\tau)E_2 + \tau E_2 e^F$$

is uniformly elliptic and we can apply Theorem 2 of [6].  $\square$

**Corollary 5.6.** *Under the same hypotheses of Theorem 5.5 we have that*

$$\mathfrak{S}_\tau \subset C^\infty(\mathbb{T}^2)$$

for all  $0 \leq \tau \leq 1$ .

*Proof.* It follows from Theorems 1 and 3 of [7].  $\square$

**Theorem 5.7.** *Under the hypotheses (5.2) to (5.7), we have that for all  $\tau \in [0, 1]$  the equation*

$$A_{11}[u]A_{22}[u] - (A_{12}[u])^2 = E_1 + (1-\tau)E_2 + \tau E_2 e^F$$

has a solution in  $C^\infty(\mathbb{T}^2)$  satisfying condition (5.8).

In particular, for  $\tau = 1$  we obtain that equation (5.1) is solvable.

*Proof.* In view of Corollary 5.6, it is sufficient to prove the existence of a  $C^2$ -solution. Hence we have to prove that  $\mathfrak{S}_\tau(\mathbb{T}^2) \neq \emptyset$  for all  $\tau \in [0, 1]$ . If  $\tau = 0$ , thanks to (5.5) we have that  $0 \in \mathfrak{S}_0$ . Then we may set

$$\rho = \sup\{\sigma \in [0, 1] \mid \mathfrak{S}_\tau \neq \emptyset, \forall \tau \in [0, \sigma]\}$$

We must show that  $\mathfrak{S}_\rho \neq \emptyset$  and that  $\rho = 1$ .

Consider a sequence  $\tau_n \in [0, \rho]$  converging to  $\rho$  and such that  $\mathfrak{S}_{\tau_n} \neq \emptyset$ . Let  $u_n \in \mathfrak{S}_{\tau_n}$  for all  $n$ . By Theorem 5.5, the sequence  $(u_n)$  is bounded in  $C^{2,1/2}(\mathbb{T}^2)$ , hence, by Ascoli-Arzelà theorem, it contains a subsequence  $(v_n)$  which converges in  $C^2(\mathbb{T}^2)$  to a function  $v$ , which is a solution belonging to  $\mathfrak{S}_\rho$ .

Now we show that  $\rho = 1$ . Assume by contradiction  $\rho < 1$  and let  $C_*^{k,1/2}(\mathbb{T}^2)$  be the space of functions  $u \in C^{k,1/2}(\mathbb{T}^2)$  satisfying  $\int_{\mathbb{T}^2} u = 0$ . Consider the map

$$T : C_*^{2,1/2}(\mathbb{T}^2) \times [0, 1] \rightarrow C_*^{0,1/2}(\mathbb{T}^2),$$

defined as

$$T(u, \tau) = A_{11}[u]A_{22}[u] - (A_{12}[u])^2 - E_1 - (1 - \tau)E_2 - \tau E_2 e^F.$$

Observe that

$$\int_{\mathbb{T}^2} T(u, \tau) = 0,$$

thanks to (5.4), (5.5) and (5.7).

We know that there exists  $v \in \mathfrak{S}_\rho \subset C_*^{2,1/2}(\mathbb{T}^2)$  such that  $T(v, \rho) = 0$ . We have that

$$T'[v, \rho](w, 0) = Lw,$$

with

$$L : C_*^{2,1/2}(\mathbb{T}^2) \rightarrow C_*^{0,1/2}(\mathbb{T}^2)$$

given by

$$\begin{aligned} (5.16) \quad Lw &= (A_{22}[v] + C_{22})w_{xx} - 2(A_{12}[v] + C_{12})w_{xy} + (A_{11}[v] + C_{11})w_{yy} \\ &+ \left( B_{11}(A_{22}[v] + C_{22}) - 2B_{12}(A_{12}[v] + C_{12}) + B_{22}(A_{11}[v] + C_{11}) \right) w_y \\ &+ D(A_{22}[v] + C_{22})w. \end{aligned}$$

Now from Lemma 5.1 and hypotheses (5.3) the matrices

$$\begin{bmatrix} A_{11}[v] & A_{12}[v] \\ A_{12}[v] & A_{22}[v] \end{bmatrix}, \quad \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}$$

are positive, so their sum is positive too, and the operator  $L$  is uniformly elliptic. Since  $D(A_{22}[v] + C_{22}) \leq 0$ , we may apply the strong maximum principle ([5], Theorem 3.5) and obtain that  $Lw = 0$  implies that  $w$  is constant, that is  $w = 0$ , by condition (5.8). Ellipticity and classical Schauder estimates ([5], Theorem 6.2) show that  $L$  is onto. Since  $L$  is one-to-one, it must be an isomorphism. Then by the implicit function theorem there



exists  $\epsilon > 0$  such that  $\mathfrak{S}_\tau(\mathbb{T}^2) \neq \emptyset$  for  $\rho < \tau < \rho + \epsilon$ , in contradiction with the definition of  $\rho$ .  $\square$

## REFERENCES

- [1] S. K. Donaldson, Two-forms on four-manifolds and elliptic equations. *Inspired by S.S. Chern*, 153–172, Nankai Tracts Math. 11, World Scientific, Hackensack NJ, 2006.
- [2] A. Fino, Y.Y. Li, S. Salamon, L. Vezzoni, The Calabi–Yau equation on 4-manifolds over 2-tori, preprint math.DG/1103.3995, to appear in *Trans. Amer. Math. Soc.*.
- [3] S. Fukuhara, K. Sakamoto, Classification of  $T^2$ -bundles over  $T^2$ , *Tokyo J. Math.* **6** (1983), 311–327.
- [4] H. Geiges, Symplectic structures on  $T^2$ -bundles over  $T^2$ , *Duke Math. J.* **67** (1992), 539–555.
- [5] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.
- [6] E. Heinz, Interior estimates for solutions of elliptic Monge–Ampère equations, *Proc. Sympos. Pure Math.*, Vol. IV, American Mathematical Society, Providence, R.I., 1961, 149–155.
- [7] L. Nirenberg, On nonlinear elliptic partial differential equations and Hölder continuity, *Comm. Pure Appl. Math.* **6** (1953), 103–156; addendum, 395.
- [8] V. Tosatti, B. Weinkove, S.T. Yau, Taming symplectic forms and the Calabi–Yau equation, *Proc. London Math. Soc.* **97** (2008), no. 2, 401–424.
- [9] V. Tosatti, B. Weinkove, The Calabi–Yau equation on the Kodaira–Thurston manifold, *J. Inst. Math. Jussieu* **10** (2011), no. 2, 437–447.
- [10] M. Ue, On the 4-dimensional Seifert fiberings with euclidean space orbifolds, in *A fête of topology*, 471–523, Academic Press, Boston, 1988.
- [11] M. Ue, Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds I, *J. Math. Soc. Japan* **42** (1990), no. 3, 511–540.
- [12] S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, *Comm. Pure Appl. Math.* **31** (1978), no. 3, 339–411.

Ernesto Buzano

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia  
 ernesto.buzano@unito.it

Anna Fino

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia  
 annamaria.fino@unito.it

Luigi Vezzoni

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia  
 luigi.vezzoni@unito.it